

# Deformations of Bundles and the Standard Model

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We modify a recently proposed heterotic model hep-th/0703210, giving three net-generations of standard model fermions, to get rid of an additional  $U(1)$  factor in the gauge group. The method employs a stable  $SU(5)$  bundle on a Calabi-Yau three-fold admitting a free involution. The bundle has to be built as a deformation of the direct sum of a stable  $SU(4)$  bundle and the trivial line bundle.

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In this note, which constitutes an addendum to [1], we propose a method to construct a model of the  $E_8 \times E_8$  heterotic string giving in four dimensions the gauge group and chiral matter content of the standard model. For this we embed an  $SU(5)$  bundle in the first  $E_8$  leading to a GUT gauge group  $SU(5)$ , which is afterwards broken by a Wilson line to the standard model gauge group. The non-simply connected Calabi-Yau threefold is obtained by modding a simply connected cover Calabi-Yau space  $X$  by a free involution. Therefore we search on  $X$  for an invariant  $SU(5)$  bundle of net-generation number  $\pm 6$  (for other constructions along these lines cf. [2], [3], [4]).

Models of this kind were recently constructed in [1] (cf. also [5]) but had an additional  $U(1)$  in the unbroken gauge group due to the specific form of the bundle

$$V_5 = V_4 \otimes \mathcal{O}_X(-\pi^*\beta) \oplus \mathcal{O}_X(4\pi^*\beta) \quad (0.1)$$

This is a polystable bundle and has structure group  $SU(4) \times U(1)_A$  (on the Lie algebra level) of  $V_5$  and therefore  $SU(5) \times U(1)_A$  as unbroken gauge group.

All the conditions, stability of the bundle, invariance under the involution, solution of the anomaly cancellation equation by having an effective five-brane class and finally the phenomenologically net-generation number were therefore essentially solved already on the level of  $V_4$ . The bundle  $V_4$  alone would give an unbroken gauge group  $SO(10)$ , which cannot be broken to the standard model gauge group by just turning on a  $\mathbf{Z}_2$  Wilson line corresponding to  $\pi_1(X/\mathbf{Z}_2)$ . Therefore in [1]  $V_4$  had to be enhanced to an  $SU(5)$  bundle by adding a line bundle (the combined conditions of stability and five-brane effectivity make a non-trivial extension for  $V_5$  problematical as explored in [1]). Then the structure (0.1) caused the additional  $U(1)_A$  in the gauge group.

So in a  $(4 + 1)$ -decomposition of the rank 5 structure group one has  $\left( \begin{array}{c|c} a & 0 \\ \hline 0 & d \end{array} \right) \in SU(5)$  where the  $U(1)_A$  is embedded as  $\left( \begin{array}{cccc|c} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ \hline & & & 1 & \\ & & & & -4 \end{array} \right)$ . Our goal is to turn on

the off-diagonal block elements to get a full, irreducible  $SU(5)$ , i.e.  $\left( \begin{array}{c|c} a & * \\ \hline * & d \end{array} \right)$ . The possibility to do this will be measured (in first order) by two  $Ext^1$ -groups, corresponding to each of the off-diagonal blocks, respectively. The process of turning on these off-diagonal terms means that the bundle  $V' = V_4 \oplus \mathcal{O}_X$  (for simplicity we set  $\beta = 0$ ) is deformed to a more generic bundle  $V$ ; conversely  $V'$  occurs as a specialization or

degeneration  $V \rightarrow V'$  where the off-diagonal terms again go to zero and one gets the reducible object. More precisely, we say a vector bundle  $V'$  deforms to a stable bundle  $V$  if there is a connected curve  $C$  and a vector bundle  $\mathcal{V}$  over  $C \times X$  such that  $V' \cong \mathcal{V}_{\{0\} \times X}$  for some point  $0 \in C$  and  $\mathcal{V}_{\{t\} \times X} \cong V$  for some other point  $t \in C$

$$\begin{array}{ccc} \mathcal{V} & V & \longrightarrow V' \\ \\ C & \begin{array}{c} | \\ \hline t \end{array} \begin{array}{c} | \\ \hline 0 \end{array} & \end{array} \quad (0.2)$$

This process will thereby cause the following changes in the structure group  $G$  of the bundles and the unbroken gauge group  $H$  of the four-dimensional low-energy observer who sees the commutator in  $E_8$  of  $G$

$$\begin{array}{c|cc} & V & \longrightarrow V' \\ \hline G & SU(5) & SU(4) \times U(1)_A \\ H & SU(5) & SU(5) \times U(1)_A \end{array} \quad (0.3)$$

For a generic choice of parameters with  $\alpha\beta \neq 0$  (where  $\alpha \in H^{11}(B)$  is a further twist class inherent in the construction, cf. [1] and below) one would find that the  $U(1)_A$ , which occurs in the structure group and in the gauge group, is anomalous and thereby gets massive by the Green-Schwarz mechanism. However the group theory of the *ad*  $E_8$  decomposition relevant here tells us that to secure the absence of exotic matter multiplets one has just to impose the condition  $\alpha\beta = 0$  (cf. [1]). Thereby the  $U(1)_A$  remains non-anomalous and remains in the light spectrum. It is this problem for which the present paper shows a way out by embedding the reducible bundle in a family of proper irreducible  $SU(5)$  bundles where the  $U(1)_A$  is therefore again massive on the compactification scale (as all other elements of  $E_8$  which are broken for the four-dimensional low-energy observer by the specific gauge background turned on on  $X$ ); for the specialization  $* \rightarrow 0$ , where the off-diagonal elements are turned off again, one would then get a restauration of four-dimensional gauge symmetry as this corresponds to  $m_{U(1)_A} \rightarrow 0$ , i.e., at this special point on the boundary of the bundle moduli space the  $U(1)_A$  returns into the light spectrum.

So, to get rid of this additional  $U(1)_A$  factor we will construct a stable holomorphic  $SU(5)$  bundle  $V_5$  by deforming the complex structure of the given polystable  $SU(5)$

bundle  $V' = V_4 \oplus \mathcal{O}_X$  (as said, for simplicity we work from now on with  $\beta = 0$ ).  $V_4$  is a stable  $SU(4)$  bundle, and  $\mathcal{O}_X$  is the trivial one-dimensional line bundle; therefore  $V'$  is a polystable bundle and solves the Donaldson-Uhlenbeck-Yau (DUY) equations. If such a deformation to a stable holomorphic bundle exist, the theorems of [6], [7] guarantee that  $V_5$  is a solution of the DUY equations, i.e., the equations of motion of the heterotic string.

In [10] (Corollary B.3) it has been shown that the direct sum of two stable vector bundles (say  $V, W$ ) of the same slope  $\mu(V) = \int c_1(V)J^2/rk(V)$  deforms to a stable vector bundle if the sum has unobstructed deformations and both spaces  $H^1(X, Hom(V, W))$  and  $H^1(X, Hom(W, V))$  do not vanish. Applied to our case we therefore have to show that  $H^1(X, Hom(V_4, \mathcal{O}_X))$  and  $H^1(X, Hom(\mathcal{O}_X, V_4))$  do not vanish and that  $V_4 \oplus \mathcal{O}_X$  has unobstructed deformations.

$H^1(X, End(V'))$  is the space of all first-order deformations of  $V'$ . The obstruction to extending a first order deformation to second (or higher) order lives in  $H^2(X, End_0(V'))$ . Thus if  $H^2(X, End_0(V')) = 0$  we can always lift to higher order, i.e., the deformations would be unobstructed. For instance, the tangent bundle  $TX$  of a  $K3$  surface has unobstructed deformations since  $H^2(X, End_0(TX)) \cong H^0(X, End_0(TX))^* = 0$ . As we are on a Calabi-Yau threefold, Serre duality shows that the dimensions of  $H^1(X, End(V'))$  and  $H^2(X, End(V'))$  are equal, so there are as many obstructions as deformations.

The vanishing of  $H^2(X, End(V'))$  is, however, only a sufficient condition for the existence of a global (in contrast to first-order) deformation. So in principle it is still possible to have global deformations although the obstruction space is non-vanishing. That this hope is not in vain is born out by the example of  $X$  being the quintic in  $\mathbf{P}^4$  and  $V' = TX \oplus \mathcal{O}_X$  (this example was considered first in [12], [13]), where nevertheless knows that a global deformation exists [10], [11].

The tangent space  $H^1(X, End(V'))$  to the deformations decomposes as follows (using the fact that  $H^1(X, \mathcal{O}_X) = 0$ )

$$H^1(X, End(V')) \cong H^1(X, End(V_4)) \oplus H^1(X, Hom(V_4, \mathcal{O}_X)) \oplus H^1(X, Hom(\mathcal{O}_X, V_4))$$

where the last two terms  $Ext^1(V_4, \mathcal{O}_X) = H^1(X, V_4^*)$  and  $Ext^1(\mathcal{O}_X, V_4) = H^1(X, V_4)$

parametrize non-trivial extensions

$$0 \rightarrow \mathcal{O}_X \rightarrow W \rightarrow V_4 \rightarrow 0 \quad (0.4)$$

$$0 \rightarrow V_4 \rightarrow W' \rightarrow \mathcal{O}_X \rightarrow 0 \quad (0.5)$$

We note first that the index theorem gives (by stability of  $V_4$  and  $\mu(V_4) = 0$  we have  $H^i(X, V_4) = 0$  for  $i = 0, 3$ ; note that also  $V^*$  is stable and has  $\mu(V^*) = 0$ )

$$\dim H^1(X, V_4) - \dim H^1(X, V_4^*) = -\frac{1}{2}c_3(V_4) \quad (0.6)$$

which for the physical relevant  $V_4$  is non-zero, so at least one of the two off-diagonal spaces is already non-vanishing. To actually prove that  $H^1(X, V_4)$  and  $H^1(X, V_4^*)$  are both non-vanishing we recall first the explicit construction of  $V_4$  from [1]. Note that whereas in [1] we considered the case  $x > 0$ ,  $\beta \neq 0$ , we consider here the case  $x < 0$  (this simplifies some arguments below) and  $\beta = 0$ .

As in [1] the rank four vector bundle  $V_4$  will be constructed as an extension

$$0 \rightarrow \pi^*E_1 \otimes \mathcal{O}_X(-D) \rightarrow V_4 \rightarrow \pi^*E_2 \otimes \mathcal{O}_X(D) \rightarrow 0 \quad (0.7)$$

where  $E_i$  are stable bundles on  $B$  and  $D = x\Sigma + \pi^*\alpha$  is a divisor in  $X$  (here  $X$  is a Calabi-Yau threefold elliptically fibered with two sections  $\sigma_i$  over  $B = \mathbf{P}^1 \times \mathbf{P}^1$ , cf. [1]; furthermore  $\Sigma = \sigma_1 + \sigma_2$  and  $F$  will denote the fiber). The argument for stability of  $V_4$  runs exactly parallel to the one given in [1]. To prove stability of  $V_4$  we first note that given zero slope stable vector bundles  $E_i$  on  $B$ , one can prove that the pullback bundles  $\pi^*E_i$  are stable on  $X$  [8], [9] for a suitable Kähler class  $J$ . Now for the zero slope bundle  $V_4$  constructed as an extension (with  $\pi^*E_i$  stable) we have two immediate conditions which are necessary for stability: first that  $\mu(\pi^*E_1 \otimes \mathcal{O}_X(-D)) < 0$  and second that  $\pi^*E_2 \otimes \mathcal{O}_X(D)$  of  $\mu(\pi^*E_2 \otimes \mathcal{O}_X(D)) > 0$  is not a subbundle of  $V_4$ , i.e., the extension is non-split. The first condition reduces to

$$DJ^2 = 2x(h-z)^2c_1^2 + 2z(2h-z)\alpha c_1 > 0 \quad (0.8)$$

(where  $c_i := \pi^*c_i(B)$ ) this implies in our case  $x < 0$  the condition

$$\alpha c_1 > 0 \quad (0.9)$$

The non-split condition can be expressed as  $\text{Ext}^1(\pi^*E_2 \otimes \mathcal{O}_X(D), \pi^*E_1 \otimes \mathcal{O}_X(-D)) = H^1(X, \mathcal{E} \otimes \mathcal{O}_X(-2D)) \neq 0$  where  $\mathcal{E} = E_1 \otimes E_2^*$ . As in [1] applying the Leray spectral

sequence to  $\pi: X \rightarrow B$  yields as sufficient condition for  $H^1(X, \mathcal{E} \otimes \mathcal{O}_X(-2D)) \neq 0$  the following condition (for  $x < 0$ )

$$\chi(B, \mathcal{E} \otimes \mathcal{O}_B(-2\alpha)) = 4 + 8\alpha^2 - 4\alpha c_1 - 2(k_1 + k_2) < 0 \quad (0.10)$$

Finally, it remains to determine the range in the Kähler cone where  $V_4$  is stable, i.e., that for any coherent subsheaf  $F$  of rank  $0 < r < 4$  we have  $\mu(F) < 0$ . Solving as in [1] the corresponding inequalities we find for the general Kähler class  $J = z\Sigma + h\pi^*c_1$  where  $z, h \in \mathbf{R}$  with  $0 < z < h$  the range (for  $x < 0$ ; here  $\zeta := h - z$ )

$$\frac{-xc_1^2}{(\alpha - xc_1)c_1}h^2 < h^2 - \zeta^2 < \frac{-xc_1^2}{(\alpha - xc_1)c_1 - 1}h^2 \quad (0.11)$$

In summary, we find  $V_4$  is stable if (0.10) and (0.11) are satisfied (the latter just fixes an appropriate range of  $z$ ).

Let us now derive sufficient conditions for  $H^1(X, V_4) \neq 0$  and  $H^1(X, V_4^*) \neq 0$ . The Leray spectral sequence gives the following exact sequence

$$0 \rightarrow H^1(B, \pi_* V_4^*) \rightarrow H^1(X, V_4^*) \rightarrow \quad (0.12)$$

Thus it suffices to show that  $H^1(B, \pi_* V_4^*) \neq 0$ . For this we apply  $\pi_*$  to the defining exact sequence of  $V_4^*$  and find the exact sequence

$$0 \rightarrow E_2^* \otimes \mathcal{O}_B(-\alpha) \otimes \pi_* \mathcal{O}_X(-x\Sigma) \rightarrow \pi_* V_4^* \rightarrow E_1^* \otimes \mathcal{O}_B(\alpha) \otimes \pi_* \mathcal{O}_X(x\Sigma) \rightarrow$$

For  $x < 0$  one has  $\pi_* \mathcal{O}_X(x\Sigma) = 0$  and finds  $E_2^* \otimes \mathcal{O}_B(-\alpha) \otimes \pi_* \mathcal{O}_X(-x\Sigma) \cong \pi_* V_4^*$ . It follows

$$H^1(B, \pi_* V_4^*) = H^1(B, E_2^* \otimes \mathcal{O}_B(-\alpha) \otimes \pi_* \mathcal{O}_X(-x\Sigma)) \quad (0.13)$$

As  $\pi_* \mathcal{O}_X(-x\Sigma) = \mathcal{O}_B \oplus \dots$  it will be sufficient to show that  $H^1(B, E_2^* \otimes \mathcal{O}_B(-\alpha)) \neq 0$ . The index theorem gives  $\chi(B, E_2^* \otimes \mathcal{O}_B(-\alpha)) = 2 + \alpha^2 - \alpha c_1 - k_2$  from which we conclude that

$$2 + \alpha^2 - \alpha c_1 - k_2 < 0 \implies H^1(B, E_2^* \otimes \mathcal{O}_B(-\alpha)) \neq 0 \implies H^1(X, V^*) \neq 0 \quad (0.14)$$

The same reasoning applied for  $H^1(X, V)$  yields

$$2 + \alpha^2 - \alpha c_1 - k_1 < 0 \implies H^1(B, E_1^* \otimes \mathcal{O}_B(-\alpha)) \neq 0 \implies H^1(X, V) \neq 0 \quad (0.15)$$

Let us now determine the physical constraints we have to impose on  $V_5$  in order to get a viable standard model compactification of the heterotic string. What concerns the invariance of the deformed bundle one can, as in [1] app. B, argue for the existence of invariant elements in the two non-trivial extension spaces (to solve both conditions simultaneously one can use reflection twists  $v \rightarrow -v$  in both fiber vector spaces of  $V_4$  and  $\mathcal{O}_X$ ). Further one has still to make sure that the deformability to first order, which we have checked, extends to a full global construction, which we assume can be done.

Note further that the characteristic classes of  $V'$  are invariant under deformations. Therefore we have  $c(V_5) = c(V')$  and a direct computation yields

$$c_2(V_5) = -2x(2\alpha - xc_1)\Sigma - 2\alpha^2 + k_1 + k_2 \quad (0.16)$$

$$\frac{c_3(V_5)}{2} = 2x(k_1 - k_2) \quad (0.17)$$

Further one has to satisfy the heterotic anomaly condition  $c_2(X) - c_2(V_5) = [W] = w_B\Sigma + a_f F$  where  $W$  is a space-time filling fivebrane wrapping a holomorphic curve of  $X$ . This leads to the condition that  $[W]$  is an effective curve class in  $X$ , which in turn can be expressed by the two conditions  $w_B \geq 0$  and  $a_f \geq 0$ . Inserting the expressions for  $c_2(X) = 6c_1\Sigma + 5c_1^2 + c_2$  and  $c_2(V)$  gives the conditions

$$w_B = (6 - 2x^2)c_1 + 4x\alpha \geq 0 \quad (0.18)$$

$$a_f = 44 + 2\alpha^2 - k_1 - k_2 \geq 0 \quad (0.19)$$

Finally, the physical net-generation number of chiral fermions, downstairs on  $X/\mathbf{Z}_2$ , is given by

$$N_{gen}^{phys} = x(k_1 - k_2) \quad (0.20)$$

To summarize, we get the following list of constraints (besides  $x < 0$ )

$$\alpha c_1 > 0 \quad (0.21)$$

$$2 + \alpha^2 - \alpha c_1 - k_i < 0, \quad \text{where } i = 1, 2 \quad (0.22)$$

$$2 + 4\alpha^2 - 2\alpha c_1 - (k_1 + k_2) < 0 \quad (0.23)$$

$$(6 - 2x^2)c_1 + 4x\alpha \geq 0 \quad (0.24)$$

$$44 + 2\alpha^2 - (k_1 + k_2) \geq 0 \quad (0.25)$$

$$x(k_1 - k_2) = \pm 3 \quad (0.26)$$

(and  $k_i \geq 8$  for  $h = \frac{1}{2}$ , cf. [1], app. B). One realizes that (0.24) entails  $x = -1$  and so

$$\alpha \leq c_1 \tag{0.27}$$

One finds that the following  $\alpha$ 's are possible (the entries in  $(p, q)$  refer to the multiples of the two generators in  $B = \mathbf{P}^1 \times \mathbf{P}^1$ ):

$$\alpha = (-1, 2), (1, 1), (1, 0), (0, 2), (1, 2), (2, 2) \tag{0.28}$$

besides interchanging the entries. For instance, one finds then for  $\alpha = (1, 1)$  that  $k_1 = 8 + i$ ,  $k_2 = 11 + i$  where  $i = 0, \dots, 14$  or  $\alpha = (1, 0)$  and  $i = 0, \dots, 12$  (besides interchanging the  $k_i$ ).

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